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Local and equatorial characterization of unit balls of subspaces of L_p, $p > 0$ and properties of the generalized cosine transform

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ABSTRACT

In this paper we show that there is no local equatorial characterization of bodies that embed in L_p in *odd* dimensions for all p not even, $0 < p < \infty$. However, bodies that embed in L_p for p odd are local equatorially characterizable provided that the dimension is even but not locally characterizable in general. This extends results given in Panina (1988) [13], Goodey and Weil (1992) [5], and Nazarov, Ryabogin and Zvavitch (2008) [12] concerning the local equatorial characterization of zonoids and intersection bodies. These results were for bodies that embed in L_1 and L_{-1} respectively.

1. Introduction

A centrally symmetric convex body K in $\mathbb{R}^n$ defines a norm on $\mathbb{R}^n$ by the gauge function or Minkowski functional given by K. That is, for all $x \in \mathbb{R}^n$,

$$\|x\|_K = \inf\{t \in \mathbb{R}\colon\ x \in tK\}.$$

We say that such a body (really the associated norm space $(\mathbb{R}^n, \|\cdot\|_K)$) embeds in L_p, $p > 0$ if there exists a linear operator $T: \mathbb{R}^n \to L_p([0,1])$ so that for all $x \in \mathbb{R}^n$, $\|x\|_K = \|Tx\|_{L_p}$. Clearly, a convex body embeds in L_2 (a Hilbert space) if and only if it is an ellipsoid. It is known that a body embeds in L_1 if and only if it is the polar body to a projection body (or equivalently a polar body to a zonoid where a zonoid is a limit of bodies produced by taking finite Minkowski sums of line segments). Also, bodies that embed in L_p, $p \geqslant 1$ can be characterized as polar bodies of so-called p-projection bodies introduced by Lutwak in [10] and [11]. And a body embeds in L_{-1} if and only if it is an intersection body where intersection bodies are defined by limits of bodies of the form of L where L is given by

$$\|x\|_L^{-1} = \mathrm{Vol}_{n-1}\big(K \cap x^{\perp}\big), \quad \text{if } x \in S^{n-1}$$

for some symmetric star body K. (For the definition of intersection bodies see [9], for more on characterizations of embeddability in L_1 and L_{-1} see [6, pp. 156 and 126–127] respectively and for characterizations of embeddability in L_p see [7] and the remark after Theorem 3.3 in [14].)

It is known that given any $p > 0$, provided one is in a high enough dimension, there exists a convex body that does not embed in L_p (in Example 2 of [7], Koldobsky shows that the cube does not embed in any L_p for $p > 0$ and $n \geqslant 3$). For $p = -1$ and $p = 1$ the existence of such a body provides a solution for the Busseman–Petti problem and for the Shephard problem respectively (see [6, Theorems 5.3, 5.4, 8.9 and 8.12] for more on these famous problems). While one would assume that not every convex body embeds in L_p, one would think that if a convex body locally "looked like" a body that embeds

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in L_p (or is *locally* L_p) then it would in fact embed in L_p. That is, if one encounters a convex body K such that for all $u \in S^{n-1}$ there exists a neighborhood $U_u \subset S^{n-1}$ of u and a centrally symmetric convex body D_u that embeds in L_p such that the boundaries of K and D_u (and so their norms) coincide at all points belonging to U_u, one would hope that therefore K also embeds in L_p. For p positive and even the answer is a definitive yes, as will be shown later. However, for $p = 1, -1$ this question spawned more thought.

It was shown by Weil in [16] that there exists no local characterization of convex bodies that embed in L_1 and much later it was shown by Nazarov, Ryabogin, and Zvavitch in [12] that no such characterization exists for convex bodies that embed in L_{-1}. Weil suggested that perhaps if one considered larger equatorial neighborhoods then one could locally characterize convex bodies that embed in L_1. That is if K is a centrally symmetric convex body such that for any equator $\sigma \subset S^{n-1}$, there exists a neighborhood E_σ of σ and a centrally symmetric convex body D_σ that embeds in L_1 such that the boundaries of K and D_σ (and so their norms) coincide at all points belonging to E_σ, is it true that K embeds in L_1? (note that since these equatorial neighborhoods are relatively large, locally characterizable implies local equatorially characterizable) This was shown to be true for even dimensions with independent solutions given by Panina in [13] and Goodey and Weil in [5] and false when the dimension is odd by Nazarov, Ryabogin, and Zvavitch in [12]. When the same question was considered for convex bodies that embed in L_{-1}, Nazarov, Ryabogin and Zvavitch also discovered the same dimensional parity dependence: convex bodies that embed in L_{-1} are local equatorially characterizable in even dimensions but not in odd dimensions.

In this paper we consider the same questions for convex bodies in $\mathbb{R}^n$ that embed in L_p, $0 < p < \infty$. We will show that for p not an integer there is no hope (see Table 1): there is no local equatorial characterization for bodies that embed in L_p for these p. Also, if p and n are both odd then there exists no local equatorial characterization for bodies that embed in L_p. That is for these instances there exists a body that is local equatorially L_p but does not embed in L_p. However if p is odd and n is even, bodies that embed in L_p are local equatorially characterizable. So for p odd we recover the same dimensional dependence that was discovered by Weil for $p = 1$ and [12] for $p = -1$.

Table 1
Capable characterizations of embeddings of convex symmetric bodies in $\mathbb{R}^n$ into L_p.

p	Parity of $n \geqslant 5$[a]	Local equatorially characterizable	Locally characterizable
-1	even	yes	no
	odd	no	
Odd, $\geqslant 1$	even	yes	no
	odd	no	
Non-integer, > 0	N/A	no	no
Even, $\geqslant 2$	N/A	yes	yes

[a] For consistency, in this table we only consider bodies in dimension 5 or higher because every convex symmetric body in dimension 4 or lower embeds in L_{-1} and every convex symmetric body in dimension 2 embeds in L_1. However, for the other values of p in the table we need only to consider $n \geqslant 2$.

Directly exhibiting an isometric isomorphism from $(\mathbb{R}^n, \|\cdot\|_K)$ to L_p is very difficult. Fortunately, an important theorem due to Levy states that a centrally symmetric convex body $K \subset \mathbb{R}^n$ embeds in L_p, $p > 0$ if and only if the norm associated with that body can be written as

$$\|x\|_K^p = \int_{S^{n-1}} |x \cdot \theta|^p \, d\mu(\theta) \tag{1}$$

where μ is some finite positive Borel measure on S^{n-1} (see [6, p. 117]). This is called the Levy representation. Note that if p is even, the absolute value around the inner product disappears and $\|\cdot\|_K^p$ becomes a polynomial in at most n variables. Since any two polynomials that agree on a neighborhood must agree everywhere one has, by a compactness argument, that if K is locally L_p with p even then K in fact embeds in L_p. For p not even exhibiting such a measure is just as difficult as exhibiting an isometric isomorphism. Another characterization of embeddability in L_p given by Koldobsky (see [6, p. 121]) provides a connection with the Fourier transform. This characterization is stated as follows:

Theorem 1.1. *$(\mathbb{R}^n, \|\cdot\|_K)$ embeds isometrically in L_p if and only if for $p > 0$ not even $\Gamma(-p/2)\widehat{(\|\cdot\|_K^p)}$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$.*

Here the Fourier transform is in the sense of distributions. Using these two characterizations and the Fourier analytic inversion formula for the p-Cosine transform, we obtain our results.

2. Auxiliary results

The main tool used for the proofs contained in this paper is the Fourier transform of distributions (see [3,4,6] for exact definitions and properties) and the connections between the p-Cosine transform and the Fourier transform.

For a complex-valued function f on $\mathbb{R}^n$, we define the p-Cosine transform or Generalized Cosine transform by the formula

$$Cos_p(f)(x) = \int_{S^{n-1}} |x \cdot \theta|^p f(\theta)\, d\theta.$$

In [15] and [14] it was proved that for $p > -1$ not even,

$$\hat{g}(\theta) = \frac{1}{4\pi C_p} Cos_p(g)(\theta) = \frac{1}{4\pi C_p} \int_{S^{n-1}} |\theta \cdot y|^p g(y)\, dy, \tag{2}$$

for all θ on the sphere where

$$C_p = \frac{2^{p+1}\sqrt{\pi}\,\Gamma((p+1)/2)}{\Gamma(-p/2)},$$

provided that g is an even homogeneous function of degree $-n-p$ on $\mathbb{R}^n \setminus 0$, $n > 1$ such that $g \in L_1(S^{n-1})$. So provided one has the correct homogeneity requirements, the p-Cosine transform is the same as the Fourier transform up to a constant. Hence, for p not even and by the uniqueness of the Fourier transform, Cos_p is linear and injective from the set of functions in $C^\infty(S^{n-1})$ to itself (see for example [6, p. 59, Lemma 3.16(ii)]). Therefore Cos_p^{-1} is linear and well defined for these functions and further, if $f \in C^\infty(S^{n-1})$ is homogeneous of degree p then by the fact that the Fourier transform is self inverting (up to constant $(2\pi)^n$ where n is the dimension), one has that

$$Cos_p^{-1} f(\theta) = \frac{1}{2(2\pi)^{n+1} C_p} \hat{f}(\theta). \tag{3}$$

In this paper, with one exception, all convex bodies will be considered to be infinitely smooth (so $\|\cdot\|_K \in C^\infty(S^{n-1})$). Using the three formulas above, this means, by the Levy representation, a C^∞ body embeds in L_p, p not even if there exists an even non-negative infinitely differentiable function f on S^{n-1} such that

$$\|x\|_K^p = \int_{S^{n-1}} |x \cdot \theta|^p f(\theta)\, d\theta.$$

So this equality, the Levy Representation, can be rewritten to say that a C^∞ body embeds in L_p, p not even if there exists an even non-negative infinitely differentiable function f on S^{n-1} such that

$$\|x\|_K^p = Cos_p(f)(x).$$

Another key ingredient is a formula connecting the Fourier transform of powers of the norm of a convex body with the derivatives of the parallel section function. We denote the $(n-1)$-dimensional Lebesgue measure in an appropriate hyperplane by $\mathrm{Vol}_{n-1}(\cdot)$. Let D be an infinitely smooth origin symmetric convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and let $\xi^\perp = \{x \in \mathbb{R}^n\colon\ x \cdot \xi = 0\}$. We denote by

$$A_{D,\xi}(t) = \mathrm{Vol}_{n-1}\big(D \cap \{\xi^\perp + t\xi\}\big), \quad t \in \mathbb{R},$$

the parallel section function of D in the direction of ξ. By Lemma 2.4 in [6], for any $m \in \mathbb{N} \cup \{0\}$, there exists an interval $(-\delta_m, \delta_m)$ so that the parallel section functions $A_{D,\xi}$ are uniformly differentiable up to the order m in this interval, and, for any t in this interval and $0 \leqslant k \leqslant m$, the functions $\xi \mapsto A_{D,\xi}^{(k)}(t)$ are continuous on S^{n-1}. The following formula was proved in [2] (see [6, p. 60]):

For any $\xi \in S^{n-1}$ and for any $k \in (-1, \infty)$, $k \neq n-1$

$$\cos(\pi k/2)\big(\|\cdot\|_D^{-n+k+1}\big)^\wedge(\xi) = \pi(n-k-1) A_{D,\xi}^{(k)}(0), \tag{4}$$

where $A_{D,\xi}^{(k)}(0)$ is the fractional derivative of order k of the parallel section function evaluated at zero given by

$$A_{D,\xi}^{(k)}(0) = \frac{1}{\Gamma(-k)} \int_0^{\delta_m} t^{-1-k}\left(A_{D,\xi}(t) - A_{D,\xi}(0) - \cdots - A_{D,\xi}^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!}\right) dt$$
$$+ \frac{1}{\Gamma(-k)} \int_{\delta_m}^{\infty} t^{-1-k} A_{D,\xi}(t)\, dt + \sum_{s=0}^{m-1} \frac{\delta_m^{s-k} A_{D,\xi}^{(s)}(0)}{s!(s-k)}, \tag{5}$$

where $m > k+1$. In particular if $k \geqslant 0$, $k \neq n-1$, is an even integer then we get the usual derivative of order k:

$$\big(\|\cdot\|_D^{-n+k+1}\big)^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{D,\xi}^{(k)}(0). \tag{6}$$

If $k \geqslant 1$, $k \neq n-1$ is an odd integer, then

$$\big(\|\cdot\|_D^{-n+k+1}\big)^\wedge(\xi) = (-1)^{(k+1)/2}2(n-k-1)k!\int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - \cdots - A_{D,\xi}^{k-1}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\,dz. \tag{7}$$

As a consequence of Eqs. (4), (6), and (7) with $k = p+n-1$ along with Koldobsky's characterization for embeddability in L_p we obtain the Fourier analytic characterization of bodies that embed in L_p.

Theorem 2.1. *Let D be an origin symmetric convex body in $\mathbb{R}^n$ such that $\|\cdot\|_D \in C^\infty(S^{n-1})$. The body D embeds in L_p, $p > 0$ if and only if for all $\xi \in S^{n-1}$*

$$\Gamma(-p/2)\frac{\pi(-p)}{\cos(\frac{\pi(p+n-1)}{2})}A_{D,\xi}^{(p+n-1)}(0) \geqslant 0 \tag{8}$$

for p not an integer where $A_{D,\xi}^{(p+n-1)}(0)$ is the fractional derivative of order $p+n-1$ evaluated at zero as defined before,

$$(-1)^{(p+n-1)/2}\Gamma(-p/2)\pi(-p)A_{D,\xi}^{(p+n-1)}(0) \geqslant 0 \tag{9}$$

for p odd and n even, and

$$(-1)^{\frac{p+n}{2}}2\Gamma(-p/2)(-p)(p+n-1)!\int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - \cdots - A_{D,\xi}^{(p+n-2)}(0)\frac{z^{p+n-2}}{(p+n-2)!}}{z^{p+n}}\,dz \geqslant 0 \tag{10}$$

for p odd and n odd.

Remark 2.2. For $p = 1$ this was noticed by Koldobsky, Ryabogin, and Zvavitch in [8].

Remark 2.3. The expressions in this theorem are precisely $\Gamma(-p/2)\widehat{(\|\cdot\|_D^p)}$. If one considers $p \in (-n, 0)$, then by removing the factor of $\Gamma(-p/2)$ in the three above expressions one gets characterizations for embeddability in L_p with p negative.

In the next section we give an example of how to use the above formulas.

3. An example of a body in $\mathbb{R}^2$ that does not embed in L_p, for odd p

The Levy representation (1) makes it a trivial exercise to create a symmetric convex body that does not embed in L_p, when p is even. Any symmetric convex body whose norm cannot be written as a polynomial will suffice. However, the fact that there exist symmetric convex bodies that do not embed in L_p when p is not even is trickier to demonstrate. For $p = 1$ this is not possible as all convex symmetric bodies in $\mathbb{R}^2$ embed in L_1 (see [6, p. 120]). For C^∞ bodies one can easily see that using Theorem 2.1 with $p = 1$ and $n = 2$ makes (9) non-negative by Brunn's theorem concerning the maximality of the central section of a convex body and the second derivative test. One needs to go to $\mathbb{R}^3$ to find symmetric convex bodies that do not embed in L_1. But for all other $p > 1$ this is possible. The example given here, modeled after the example given in [6, p. 74], will be for p odd. For non-integer p other suitable examples can be constructed using the same technique with minor modifications.

Let p be odd and $n = 2$. By Koldobsky's characterization we need to find a body D such that $\Gamma(-p/2)\widehat{(\|\cdot\|_D^p)}$ is not a positive distribution on $\mathbb{R}^2 \setminus \{0\}$. Because $\Gamma(\frac{-p}{2})(-1)^{\frac{p+1}{2}}\pi(-p)$ is always negative for p odd, constructing a body D such that $A_{D,\xi}^{(p+1)}(0) > 0$ for some $\xi \in S^1$ will suffice by way of Theorem 2.1 formula (9).

Consider the even function $f(x) = 1 - x^2 + x^{p+1}$. Since $f(0) > 0$ and $f''(0) < 0$ by continuity there exists $\epsilon > 0$ such that $f > 0$ and $f'' < 0$ on $[-\epsilon, \epsilon]$. Define the body D by:

$$D = \big\{(x, y)\colon\ |y| \leqslant f(x),\ x \in [-\epsilon, \epsilon]\big\}$$

D is clearly a symmetric convex body. Also if $\xi = (1, 0)$ then $A_{D,\xi}(t) = 2(1 - t^2 + t^{p+1})$ on its support $[-\epsilon, \epsilon]$ and $A_{D,\xi}^{(p+1)}(0) = (p+1)! > 0$. Thus D does not embed in L_p.

Remark 3.1. Theorem 2.1 can only be applied if the body under consideration is infinitely smooth and D is not smooth at the points $(\pm\epsilon, f(\pm\epsilon))$, $(\pm\epsilon, -f(\pm\epsilon))$. However since this example only requires knowledge of the sections perpendicular to $\xi = (1,0)$ near the origin, one can smooth the body at these points and get a new infinitely smooth body having the same sections as D in the direction of $\xi = (1,0)$ near the origin.

Remark 3.2. By considering a new body, $D \times [-1,1]$ one has a convex body (the cylinder generated by D) in $\mathbb{R}^3$ that does not embed in L_p for p odd as the norm space generated by this new body has a subspace that does not embed in L_p. By repeatedly using this technique, one can create a body in any finite dimension that does not embed in L_p for p odd.

4. A motivating example

The following example gives motivation as to why there should be no local characterization of symmetric convex bodies that embed in L_p.

Consider the following question: *If $f \in L_1([-\pi,\pi])$ is a real-valued, even, 2π-periodic function that "locally" has non-negative Fourier coefficients defined by*

$$\hat{f}(j) = \int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt, \quad \text{for each } j \in \mathbb{Z},$$

then is it true that f has non-negative Fourier coefficients? That is if f is such that for all $x \in [-\pi,\pi]$ there exists $\epsilon > 0$ and a 2π-periodic, real-valued, even function $g_x \in L_1([-\pi,\pi])$ such that

- $\hat{g}_x(j) \geqslant 0$ for all $j \in \mathbb{Z}$,
- $g_x \equiv f$ on $(x-\epsilon, x+\epsilon)$

then must it be true that $\hat{f}(j) \geqslant 0$ for all $j \in \mathbb{Z}$? One can easily see the parallel between this question and our aforementioned question concerning bodies that embed in L_p. The answer to this question is no. There exists a 2π-periodic real-valued, even function $f \in L_1([-\pi,\pi])$ that "locally" has non-negative Fourier coefficients but actually has at least one negative Fourier coefficient.

The idea for this construction was provided by Fedor Nazarov and communicated to us by Dmitry Ryabogin. The idea is to start with a 2π-periodic, even, real-valued function g that has all Fourier coefficients being strictly positive then to perturb this function to get a new function f that has at least one negative Fourier coefficient and so that f mostly agrees with the original function g.

Consider the function given by

$$g(t) = \begin{cases} 0 & \text{if } -\pi \leqslant t \leqslant -\frac{\pi}{4}, \\ \frac{4}{\pi}t + 2 & \text{if } -\frac{\pi}{4} \leqslant t \leqslant 0, \\ -\frac{4}{\pi}t + 2 & \text{if } 0 \leqslant t \leqslant \frac{\pi}{4}, \\ 0 & \text{if } \frac{\pi}{4} \leqslant t \leqslant \pi. \end{cases}$$

All of the Fourier coefficients for this function are strictly positive. Now consider the perturbation of g given by the function f as follows:

$$f(t) = \begin{cases} \frac{4}{\pi}t + 2 & \text{if } -\pi \leqslant t \leqslant 0, \\ -\frac{4}{\pi}t + 2 & \text{if } 0 \leqslant t \leqslant \pi. \end{cases}$$

For each $x \in [0,\pi]$, $\epsilon > 0$ define the following functions $g_{x,\epsilon}$:

$$g_{x,\epsilon}(t) = \begin{cases} f(t)\mathbf{1}_{(-\frac{\pi}{4}-\epsilon, \frac{\pi}{4}+\epsilon)} & \text{if } x \in [0, \frac{\pi}{4}], \\ g(t) + f(t)\mathbf{1}_{(-x-\epsilon,-x+\epsilon)\cup(x-\epsilon,x+\epsilon)} & \text{if } x \in (\frac{\pi}{4}, \pi), \\ g(t) + f(t)\mathbf{1}_{[-\pi,-\pi+\epsilon)\cup(\pi-\epsilon,\pi]} & \text{if } x \in \{\pi\}. \end{cases}$$

Since each $g_{x,\epsilon}$ is even, for $x \in [-\pi,0]$ define $g_{x,\epsilon} = g_{-x,\epsilon}$. Note that for each $x \in [-\pi,\pi]$ and for ϵ small, $g_{x,\epsilon} \equiv g$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Also note that f agrees with $g_{x,\epsilon}$ on $(-x-\epsilon, -x+\epsilon) \cup (x-\epsilon, x+\epsilon)$ for small enough ϵ. Hence, for each $x \in [-\pi,\pi]$ by the linearity of the integral and the strict positivity of the Fourier coefficients of g, there exists ϵ small enough so that

- $\hat{g}_{x,\epsilon}(j) \geqslant 0$ for all $j \in \mathbb{Z}$,
- $g_{x,\epsilon} \equiv f$ on $(-x-\epsilon, -x+\epsilon) \cup (x-\epsilon, x+\epsilon)$

are both true. However, $\hat{f}(0) = \int_{-\pi}^{\pi} f(t)\,dt < 0$. So we've just constructed a 2π-periodic integrable function f that "locally" has non-negative Fourier coefficients, yet has a negative Fourier coefficient.

5. There is no local equatorial characterization of bodies that embed in L_p for $p \geqslant 1$ odd in odd dimensions

In this section assume that $p \geqslant 1$ is odd and the dimension is odd and larger than one. To construct a counterexample, it is natural to use (10). This formula shows that one has to use the information about the section function $A_{D,\xi}(z)$ of the body along the whole range of z.

For $0 < \varepsilon < 1$ and $\xi \in S^{n-1}$, we denote by $U_\varepsilon(\xi)$ the union of caps centered at ξ and $-\xi$:

$$U_\varepsilon(\xi) := \left\{\theta \in S^{n-1}\colon |\theta \cdot \xi| \geqslant \sqrt{1-\varepsilon^2}\right\}.$$

We denote by $E_\varepsilon(\xi)$, $0 < \varepsilon < 1$, the neighborhood of the equator $S^{n-1} \cap \xi^\perp$:

$$E_\varepsilon(\xi) := \left\{\theta \in S^{n-1}\colon |\theta \cdot \xi| < \varepsilon\right\}.$$

The following result is crucial for the construction of the counterexample. Its proof is based on the fact that the inversion formula (3) with (10) is not local.

Lemma 5.1. *Let $n \geqslant 3$ be odd and $p \geqslant 1$ be odd and fixed. Then there exist an $\varepsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \xi \in S^{n-1}$, there exists an even function $f_{x,\xi} \in C^\infty(S^{n-1})$ satisfying $f_{x,\xi} = 0$ on $E_\varepsilon(x)$, and $Cos_p^{-1} f_{x,\xi} \geqslant c$ on $U_\varepsilon(\xi)$.*

Proof. First, we fix $x, \xi \in S^{n-1}$ and find $\varepsilon = \varepsilon(x,\xi)$ and $c = c(x,\xi)$ satisfying the requirement of the lemma. Then we use a compactness argument to produce absolute ε and c.

For fixed $x, \xi \in S^{n-1}$ and some small $\varepsilon > 0$ there exist two infinitely smooth symmetric convex bodies M and Q such that $\|\cdot\|_M = \|\cdot\|_Q$ on the closure of $E_\varepsilon(\xi) \cup E_\varepsilon(x)$, and $\|\cdot\|_Q > \|\cdot\|_M$ otherwise. Set $f_{x,\xi}(\cdot) = k_p(\|\cdot\|_Q^p - \|\cdot\|_M^p)$ where k_p is either 1 or -1. Then $f_{x,\xi}$ is an even infinitely differentiable function such that $f_{x,\xi} = 0$ on $E_\varepsilon(x)$. Also $\|\cdot\|_M = \|\cdot\|_Q$ on $E_\varepsilon(\xi)$ implies $A^{(k)}_{M,\xi}(0) = A^{(k)}_{Q,\xi}(0)$, $k = 0, 1, \dots, p+n-2$ because differentiability at zero is a local property. Thus, the linearity of Cos_p^{-1}, (3) and (10) imply

$$\begin{aligned} Cos_p^{-1} f_{x,\xi}(\xi) &= k_p\left(Cos_p^{-1}\left(\|\cdot\|_Q^p\right)(\xi) - Cos_p^{-1}\left(\|\cdot\|_M^p\right)(\xi)\right) \\ &= \frac{k_p}{2(2\pi)^{n+1} C_p}\left(\widehat{\|\cdot\|_Q^p}(\xi) - \widehat{\|\cdot\|_M^p}(\xi)\right) \\ &= \frac{k_p(-1)^{\frac{p+n}{2}} 2\Gamma(-p/2)(-p)(p+n-1)!}{2(2\pi)^{n+1} C_p} \int_0^\infty \frac{A_{Q,\xi}(z) - A_{M,\xi}(z)}{z^{p+n}}\, dz. \end{aligned} \tag{11}$$

By the choice of k_p the fraction in the last line is strictly negative. Also $A_{Q,\xi} = A_{M,\xi}$ near zero and $A_{Q,\xi} < A_{M,\xi}$ elsewhere on the union of their supports because the boundaries of Q and M agree on $E_\varepsilon(\xi)$ and $Q \subsetneq M$. Hence the integral is strictly negative making the last line strictly positive. So, we have exhibited that for fixed $x, \xi \in S^{n-1}$ there exists $\varepsilon' = \varepsilon'(x,\xi) > 0$ and $c' = c'(x,\xi)$ such that there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $E_\varepsilon(x)$, and $Cos_p^{-1} f_{x,\xi}(\xi) \geqslant c'$.

The function $Cos_p^{-1} f_{x,\xi}$ is continuous on S^{n-1} since M, Q are infinitely smooth (see [6, Lemma 2.4]). Hence, $Cos_p^{-1} f_{x,\xi} \geqslant c > 0$ on $U_{\varepsilon''}(\xi)$, for some $\varepsilon'' > 0$ and $c = c(x,\xi)$. Put $\tilde\varepsilon = \tilde\varepsilon(x,\xi) = \min(\varepsilon', \varepsilon'')$. We proved that for any x and ξ, there is $\tilde\varepsilon = \tilde\varepsilon(x,\xi) > 0$ and a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $E_{\tilde\varepsilon}(x)$, but $Cos_p^{-1} f_{x,\xi} \geqslant c$ on $U_{\tilde\varepsilon}(\xi)$, $c = c(x,\xi)$.

Now we use the compactness argument to show that we can choose ε and c independent of x and ξ. We choose a finite set of pairs $\{x_i, \xi_i\}_{i=1}^m$ such that $\{U_{\tilde\varepsilon_i/2}(x_i) \times U_{\tilde\varepsilon_i/2}(\xi_i)\}_{i=1}^m$ cover $S^{n-1} \times S^{n-1}$. We take

$$\varepsilon = \frac{1}{2} \min_{1 \leqslant i \leqslant m} \tilde\varepsilon_i \quad \text{and} \quad c = \min_{1 \leqslant i \leqslant m} c(x_i, \xi_i).$$

Then, for any (x,ξ), there is a pair (x_i, ξ_i) such that $(x,\xi) \in U_{\tilde\varepsilon_i/2}(x_i) \times U_{\tilde\varepsilon_i/2}(\xi_i)$ and thereby

$$E_\varepsilon(x) \times U_\varepsilon(\xi) \subset E_{\tilde\varepsilon_i}(x_i) \times U_{\tilde\varepsilon_i}(\xi_i).$$

Finally, we may define $f_{x,\xi} = f_{x_i,\xi_i}$. $\square$

Remark 5.2. The constant k_p is given by $-sgn\left(\frac{(-1)^{(p+n)/2} 2\Gamma(-p/2)(-p)(p+n-1)!}{C_p}\right)$.

Remark 5.3. Note that, by dilating M and Q simultaneously (and thus the functions $f_{x,\xi}$), we may assume that c is as large as we want. For technical reasons that will become clear later, we take $c = 2Cos_p^{-1}(\|\cdot\|_{B_2^n}^p) = 2Cos_p^{-1}\mathbf{1}$. Moreover by the compactness argument in the lemma, we obtain for free that the set of functions $\{f_{x,\xi}\}_{x,\xi \in S^{n-1}}$ in the lemma is finite.

Let C_+^∞ be the class of origin-symmetric convex bodies with C^∞ boundary and everywhere positive Gaussian curvature (see [1, p. 25]). The following lemma is similar to Lemma 3.3 in [12]. The only modifications to the proof given in [12] are the replacement of the Radon transform by the p-Cosine transform and the radial function by norm to the power p so the proof is omitted.

Lemma 5.4. *Let $M \in C_+^\infty$ and let $K(t) = tB_2^n + (1-t)M$ where the summation is in the sense of Minkowski and $t \in [0,1]$. Then the map $t \to Cos_p^{-1}(\|\cdot\|_{K(t)}^p)(\xi)$, $\xi \in S^{n-1}$, $p > -1$ is continuous.*

Lemma 5.5. *Let $n \geqslant 3$. For any point $\xi_0 \in S^{n-1}$ there exists $\widetilde{K} \in C_+^\infty$ such that $Cos_p^{-1}(\|\cdot\|_{\widetilde{K}}^p)(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $Cos_p^{-1}(\|\cdot\|_{\widetilde{K}}^p)(\pm\xi_0) = 0$.*

Proof. Fix $n \geqslant 3$. By Remark 3.2, there exists a convex symmetric body in $\mathbb{R}^n$ that does not embed in L_p. Then there exists, if necessary by approximation, $M \in C_+^\infty$ such that $Cos_p^{-1}(\|\cdot\|_M^p)(\xi)$ is sign-changing.

For $t \in [0,1]$, define a new convex body $K(t) = tB_2^n + (1-t)M$ which is also in C_+^∞. Then $Cos_p^{-1}(\|\cdot\|_{K(0)}^p)(\xi)$ is sign-changing and there exists $\Lambda' \subset S^{n-1}$ such that $Cos_p^{-1}(\|\cdot\|_{K(0)}^p)(\xi) < 0$, for all $\xi \in \Lambda'$. On the other hand, $Cos_p^{-1}(\|\cdot\|_{K(1)}^p)(\xi) > 0$, for all $\xi \in S^{n-1}$. By the previous lemma, the map $t \to Cos_p^{-1}(\|\cdot\|_{K(t)}^p)(\xi)$ is continuous. Hence by a connectedness argument, one can find an intermediate body that barely embeds in L_p. That is there exists $t_0 \in [0,1]$ such that

$$Cos_p^{-1}\big(\|\cdot\|_{K(t_0)}^p\big)(\xi) \geqslant 0, \quad \forall \xi \in S^{n-1} \quad \text{and} \quad Cos_p^{-1}\big(\|\cdot\|_{K(t_0)}^p\big)(\xi) = 0, \quad \forall \xi \in \Lambda \subset S^{n-1},$$

for some $\Lambda \neq \emptyset$. Fix any $\xi_0 \in \Lambda$. Consider an even C^∞ smooth function g on S^{n-1} such that

$$g(x) > 0, \quad \forall x \neq \pm\xi_0 \quad \text{and} \quad g(\pm\xi_0) = 0.$$

For $\varepsilon > 0$ define a body $\widetilde{K}$ (depending on ξ_0) by

$$Cos_p^{-1}\big(\|\cdot\|_{\widetilde{K}}^p\big)(\xi) = Cos_p^{-1}\big(\|\cdot\|_{K(t_0)}^p\big)(\xi) + \varepsilon g(\xi).$$

Note that $Cos_p^{-1}(\|\cdot\|_{\widetilde{K}}^p)(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $Cos_p^{-1}(\|\cdot\|_{\widetilde{K}}^p)(\pm\xi_0) = 0$. We get

$$\|x\|_{\widetilde{K}}^p = \|x\|_{K(t_0)}^p + \varepsilon\, Cos_p(g)(x).$$

Since $Cos_p(g)$ is a C^∞ function, and $K(t_0) \in C_+^\infty$, we may choose ε small enough so that $\widetilde{K} \in C_+^\infty$. Using a rotation argument, we can take ξ_0 to be arbitrary. □

Remark 5.6. For $p \geqslant 1$ we can dispense with the use of the technical Lemma 5.4 in Lemma 5.5 by considering Minkowski p-summation as for $p \geqslant 1$, convexity is maintained by this summation. If we consider for $t \in [0,1]$ the body $K(t)$ defined by $\|\cdot\|_{K(t)}^p = t\|\cdot\|_{B_2^n}^p + (1-t)\|\cdot\|_M^p$, then $K(t) \in C^\infty(S^{n-1})$, and also $K(t)$ is convex as $p \geqslant 1$ makes $\|\cdot\|_{K(t)}$ a norm. The map $t \to Cos_p^{-1}(\|\cdot\|_{K(t)}^p)(\xi)$ is now continuous in the sup norm on $C(S^{n-1})$ by the linearity of Cos_p^{-1}. The rest of Lemma 5.5 follows.

Theorem 5.7. *Let $n \geqslant 3$ be odd. There exists $\varepsilon > 0$ and a convex symmetric body K that does not embed in L_p, but nevertheless for all $x \in S^{n-1}$ there exists a body L_x that embeds in L_p, $p > 0$ such that $\|\cdot\|_K = \|\cdot\|_{L_x}$ on $E_\varepsilon(x)$.*

Proof. We define a convex body K and a family of convex bodies $\{L_x\}_{x \in S^{n-1}}$ using $\widetilde{K}$ from Lemma 5.5 and the functions f_{x,ξ_0} from Lemma 5.1. We fix some small ε satisfying the requirements of Lemma 5.1 and we may assume that $c = 2Cos_p^{-1}(\|\cdot\|_{B_2^n}^p)$ (see Remark 5.3). Then, define $K = K_{\delta,\xi_0}$ via $\|\cdot\|_K^p = \|\cdot\|_{\widetilde{K}}^p - \delta\|\cdot\|_{B_2^n}^p$, where for the moment $\delta > 0$ is assumed to be so small that $K \in C_+^\infty$ and $Cos_p^{-1}(\|\cdot\|_K^p)$ is strictly positive outside $U_\varepsilon(\xi_0)$. Note that $Cos_p^{-1}(\|\cdot\|_K^p)(\xi_0) < 0$ and thus K does not embed in L_p.

Now we define a family of convex bodies $\{L_x\}_{x \in S^{n-1}}$. Since $\widetilde{K} \in C_+^\infty$, we take δ so small that $\|\cdot\|_{L_x}^p := \|\cdot\|_{\widetilde{K}}^p - \delta\|\cdot\|_{B_2^n}^p + \delta f_{x,\xi_0} > 0$ on S^{n-1} and L_x is convex. Observe that $\|\cdot\|_{L_x} = \|\cdot\|_K$ on $E_\varepsilon(x)$ for any $x \in S^{n-1}$.

We can assume that δ is so small that

$$Cos_p^{-1}\big(\|\cdot\|_{L_x}^p\big) = Cos_p^{-1}\big(\|\cdot\|_{\widetilde{K}}^p\big) - \delta Cos_p^{-1}\big(\|\cdot\|_{B_2^n}^p\big) + \delta Cos_p^{-1} f_{x,\xi_0} > 0$$

on $S^{n-1} \setminus U_\varepsilon(\xi_0)$, since $Cos_p^{-1}(\|\cdot\|_{\widetilde{K}}^p) > 0$ on $S^{n-1} \setminus U_\varepsilon(\xi_0)$.

To show that bodies L_x embed in L_p for all $x \in S^{n-1}$, it is enough to prove that $Cos_p^{-1}(\|\cdot\|_{L_x}^p) > 0$ on $U_\varepsilon(\xi_0)$. By Remark 5.3, $\min_{x \in S^{n-1}} Cos_p^{-1} f_{x,\xi_0} \geqslant 2Cos_p^{-1}(\|\cdot\|_{B_2^n}^p)$ on $U_\varepsilon(\xi_0)$, hence

$$Cos_p^{-1}\big(\|\cdot\|_{L_x}^p\big) = Cos_p^{-1}\big(\|\cdot\|_{\widetilde{K}}^p\big) - \delta Cos_p^{-1}\big(\|\cdot\|_{B_2^n}^p\big) + \delta Cos_p^{-1} f_{x,\xi_0} \geqslant \delta Cos_p^{-1}\big(\|\cdot\|_{B_2^n}^p\big) > 0$$

on $U_\varepsilon(\xi_0)$. Moreover, $\delta > 0$ can be chosen independently of x since the set of functions $\{f_{x,\xi}\}_{x,\xi\in S^{n-1}}$ in Lemma 5.1 is finite. $\square$

Remark 5.8. If one looks back, one can see the parallels between the result in Theorem 5.7 concerning bodies that embed in L_p and the solution to the question asked in Section 4 concerning Fourier coefficients. The body $\widetilde{K}$ is the analogue of the function $g(t)$ in Section 4, the bodies L_x fill in for the functions $g_{\epsilon,x}(t)$ and the body K corresponds to the function $f(t)$ from Section 4.

6. There is no local equatorial characterization of bodies that embed in L_p for $p > 0$ not an integer in any dimension bigger than 1

The proof of this statement is exactly the same as in the previous section. The only modifications are the fact that everything can take place in dimension 2 or higher and some minor modifications in Lemma 5.1. In this lemma we define the $f_{x,\xi}$ in nearly the same way. We only change k_p accordingly and use Theorem 2.1 with formula (8) and necessarily formula (5). However, (5) still requires knowledge about the section function along its entire support. This allows us to perform nearly the same computation as in (11). The remainder of the lemmas and the theorem remain unchanged.

7. There is a local equatorial characterization of bodies that embed in L_p for odd p, $p \geqslant 1$ in even dimensions

The proof of the following lemma is obtained by a straightforward repetition of the argument from [6, p. 60], and we omit the details.

Lemma 7.1. *Let $g(x)$ be a homogeneous function of degree p such that $g(x)$ is non-negative and infinitely smooth on S^{n-1}. Then*

$$\hat{g}(\xi) = (-1)^{(p+n-1)/2}\pi(-p)A_{g,\xi}^{(p+n-1)}(0),$$

where

$$A_{g,\xi}(z) = \int\limits_{\{y\in\mathbb{R}^n:\ y\cdot\xi=z\}} \chi_{[0,1]}\big(\sqrt[p]{g(y)}\big)\,dy, \quad \xi\in S^{n-1}.$$

Also, the following powerful lemma is well known (see for example Lemma 1 in [7]).

Lemma 7.2. *Let $p > 0$ not even, $\xi\in\mathbb{R}^n$, $\xi\neq 0$. Then for every test function with 0 not in $\operatorname{supp}\phi$, one has*

$$\int\limits_{\mathbb{R}^n} |\xi\cdot x|^p\hat{\phi}(x)\,dx = (2\pi)^{n-1}C_p\int\limits_{\mathbb{R}} |t|^{-1-p}\phi(t\xi)\,dt.$$

Using this lemma we show the following.

Lemma 7.3. *Let ϕ be a non-negative test function on $\mathbb{R}^n$ with support outside the origin. Then the function*

$$\alpha(x) := \int\limits_0^\infty r^{n+p-1}\hat{\phi}(rx)\,dr$$

is homogeneous of degree $-n-p$ and infinitely smooth on $\mathbb{R}^n\setminus\{0\}$ and the function

$$g(x) := \frac{1}{2(2\pi)^2}\int\limits_{\mathbb{R}} |t|^{-1-p}\phi(tx)\,dt$$

is homogeneous of degree p and infinitely smooth on $\mathbb{R}^n$ with $\hat{g}(\xi)=\alpha(\xi)$, $\forall\xi\in S^{n-1}$.

Proof. Using (2) and Lemma 7.2 we get for all $\xi\in S^{n-1}$

$$\hat{\alpha}(\xi) = \frac{1}{4\pi C_p}\int\limits_{S^{n-1}} |\xi\cdot\theta|^p\int\limits_0^\infty r^{n+p-1}\hat{\phi}(r\theta)\,dr\,d\theta = \frac{1}{4\pi C_p}\int\limits_{S^{n-1}}\int\limits_0^\infty |\xi\cdot\theta|^p r^{n+p-1}\hat{\phi}(r\theta)\,dr\,d\theta$$

$$= \frac{1}{4\pi C_p}\int\limits_{S^{n-1}}\int\limits_0^\infty |\xi\cdot r\theta|^p\hat{\phi}(r\theta)r^{n-1}\,dr\,d\theta = \frac{1}{4\pi C_p}\int\limits_{\mathbb{R}^n} |\xi\cdot x|^p\hat{\phi}(x)\,dx$$

$$= \frac{(2\pi)^{n-1}C_p}{4\pi C_p} \int_{\mathbb{R}} |t|^{-1-p}\phi(t\xi)\,dt = (2\pi)^{n-2}\frac{1}{2}\int_{\mathbb{R}} |t|^{-1-p}\phi(t\xi)\,dt$$
$$= (2\pi)^n g(\xi).$$

Now by inverting the Fourier transform we get

$$(2\pi)^n\alpha(\xi) = \hat{\hat{\alpha}}(\xi) = (2\pi)^n\hat{g}(\xi)$$

or $\hat{g}(\xi) = \alpha(\xi)$. $\square$

Theorem 7.4. *Let n be even and let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Assume that for any great sphere $\xi^\perp \cap S^{n-1}$, there exists a body L_ξ that embeds in L_p and a neighborhood $E_{\varepsilon(\xi)}(\xi)$ of $\xi^\perp \cap S^{n-1}$ such that the Minkowski functionals of K and L_ξ coincide at all points of $E_{\varepsilon(\xi)}(\xi)$; then K embeds in L_p.*

Proof. In the case where K and each L_ξ are infinitely smooth, observe that $\|x\|_K = \|x\|_{L_\xi}$ for all $x \in E_{\varepsilon(\xi)}(\xi)$ implies $A_{K,\xi}(t) = A_{L_\xi,\xi}(t)$ for sufficiently small t. Therefore this last equality is true for all of their derivatives evaluated at zero and so for all ξ,

$$(-1)^{(p+n-1)/2}\Gamma(-p/2)\pi(-p)A_{K,\xi}^{(p+n-1)}(0) = (-1)^{(p+n-1)/2}\Gamma(-p/2)\pi(-p)A_{L_\xi,\xi}^{(p+n-1)}(0).$$

Since each L_ξ embeds in L_p, by Theorem 2.1 formula (9), the left side of this equality is non-negative. Therefore the right side is non-negative as well, and since Theorem 2.1 is biconditional, we have that K embeds in L_p.

Consider the general case. By Koldobsky's characterization Theorem 1.1 we need to show that $\Gamma(-p/2)\widehat{(\|\cdot\|_K^p)}$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$. Thus we need to show that

$$\left\langle \Gamma(-p/2)\widehat{\left(\|\cdot\|_K^p\right)}, \phi\right\rangle \geqslant 0,$$

for all non-negative test functions on $\mathbb{R}^n$ with support outside the origin. Using the definition of the Fourier transform of distributions, and passing to polar coordinates we get

$$\begin{aligned}\left\langle \Gamma(-p/2)\widehat{\left(\|\cdot\|_K^p\right)}, \phi\right\rangle &= \left\langle \Gamma(-p/2)\|\cdot\|_K^p, \hat{\phi}\right\rangle \\ &= \Gamma(-p/2)\int_{\mathbb{R}^n} \|x\|_K^p\hat{\phi}(x)\,dx \\ &= \Gamma(-p/2)\int_{S^{n-1}}\int_0^\infty \|r\theta\|_K^p\hat{\phi}(r\theta)r^{n-1}\,dr\,d\theta \\ &= \Gamma(-p/2)\int_{S^{n-1}} \|\theta\|_K^p\int_0^\infty r^{n+p-1}\hat{\phi}(r\theta)\,dr\,d\theta. \end{aligned} \tag{12}$$

Observe that the function $\alpha(x) := \int_0^\infty r^{n+p-1}\hat{\phi}(rx)\,dr$ defined on $\mathbb{R}^n \setminus \{0\}$ is homogeneous of degree $-n-p$ and infinitely smooth. By Lemma 7.3, there exists an infinitely smooth non-negative homogeneous of degree p function

$$g(x) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}} |t|^{-p-1}\phi(tx)\,dt \quad \text{such that } \hat{g}(\theta) = \alpha(\theta), \quad \forall\theta \in S^{n-1}.$$

Thus,

$$\int_{S^{n-1}} \|\theta\|_K^p\int_0^\infty r^{n+p-1}\hat{\phi}(r\theta)\,dr\,d\theta = \int_{S^{n-1}} \|\theta\|_K^p\hat{g}(\theta)\,d\theta. \tag{13}$$

Using a partition of unity on S^{n-1}, we can write

$$g(\theta) = \sum_{j=1}^m g_j(\theta) = \sum_{j=1}^m \frac{1}{2(2\pi)^2}\int_{\mathbb{R}} |t|^{-p-1}\phi_j(t\theta)\,dt, \quad \theta \in S^{n-1}, \tag{14}$$

where each ϕ_j is a non-negative test function and $\operatorname{supp} g_j|_{S^{n-1}} \subset U_{\varepsilon_j}(\xi_j)$ are small enough. Now since p is odd and n is even, the fractional derivative in Lemma 7.1 is an actual derivative. Hence $\operatorname{supp} g_j|_{S^{n-1}} \subset U_{\varepsilon_j}(\xi_j)$ implies $\operatorname{supp}\hat{g}_j|_{S^{n-1}} \subset E_{\varepsilon_j}(\xi_j)$, and (12) becomes

$$\begin{aligned}
\Gamma(-p/2)\int_{S^{n-1}}\|\theta\|_K^p\int_0^\infty r^{n+p-1}\hat{\phi}(r\theta)\,dr\,d\theta &= \Gamma(-p/2)\int_{S^{n-1}}\|\theta\|_K^p\hat{g}(\theta)\,d\theta \\
&= \Gamma(-p/2)\int_{S^{n-1}}\|\theta\|_K^p\left(\sum_{j=1}^m g_j\right)^{\wedge}(\theta)\,d\theta \\
&= \Gamma(-p/2)\sum_{j=1}^m\int_{S^{n-1}}\|\theta\|_K^p\hat{g}_j(\theta)\,d\theta \\
&= \Gamma(-p/2)\sum_{j=1}^m\int_{E_{\varepsilon_j}(\xi_j)}\|\theta\|_K^p\hat{g}_j(\theta)\,d\theta \\
&= \Gamma(-p/2)\sum_{j=1}^m\int_{E_{\varepsilon_j}(\xi_j)}\|\theta\|_{L_{\xi_j}}^p\hat{g}_j(\theta)\,d\theta \\
&= \Gamma(-p/2)\sum_{j=1}^m\int_{S^{n-1}}\|\theta\|_{L_{\xi_j}}^p\hat{g}_j(\theta)\,d\theta \\
&= \Gamma(-p/2)\sum_{j=1}^m\int_{S^{n-1}}\|\theta\|_{L_{\xi_j}}^p\int_0^\infty r^{n+p-1}\hat{\phi}_j(r\theta)\,dr\,d\theta \\
&= \sum_{j=1}^m\bigl\langle\Gamma(-p/2)\|\cdot\|_{L_{\xi_j}}^p,\hat{\phi}_j\bigr\rangle \\
&= \sum_{j=1}^m\bigl\langle\Gamma(-p/2)\bigl(\widehat{\|\cdot\|_{L_{\xi_j}}^p}\bigr),\phi_j\bigr\rangle.
\end{aligned}$$

Since each L_{ξ_j} embeds in L_p, by Theorem 1.1 each term in the final sum is non-negative making the entire sum non-negative. Hence $\langle\Gamma(-p/2)(\widehat{\|\cdot\|_K^p}),\phi\rangle$ is non-negative and K embeds in L_p. □

8. There is no local characterization of bodies that embed in L_p, p not even

In this section we prove the analog of the result of Weil [16] for zonoids. Our proof is different from the one of Weil but is extremely similar to the proof in [12]. We show that, given $x,\xi\in S^{n-1}$, one can construct a function f which is zero around x, but such that the inverse p-Cosine transform of f is positive around ξ. Based on the negative results in sections five and six, if the dimension is odd or if p is not an integer, since there is no local equatorial characterization of bodies that embed in L_p, there cannot be a local characterization of bodies that embed in L_p. Also based on previous remarks, we know the answer is affirmative for p even. So in this section we need only to consider the case where p is odd and the dimension n is even. For convenience of the reader we split the proof of this needed result (compare Lemma 8.5 with Lemma 5.1) into four statements. We will use the following notation

$$\Im_{\varepsilon,x}=\bigl\{f\in C^\infty\bigl(S^{n-1}\bigr)\text{ even: } f=0\text{ on }U_\varepsilon(x)\bigr\},\quad 0<\varepsilon<1.$$

Lemma 8.1. *Cos_p^{-1} commutes with rotations. That is $Cos_p^{-1}(f\circ\rho)\equiv(Cos_p^{-1}f)\circ\rho$ for all $f\in C^\infty(S^{n-1})$ and for all $\rho\in SO(n)$.*

Lemma 8.2. *Let $n\geqslant 3$, and let $\xi,x\in S^{n-1}$ be two orthogonal vectors. Assume that any $f\in\Im_{1/4,x}$ satisfies $Cos_p^{-1}f(\xi)=0$. Then for any pair of orthogonal vectors $u,v\in S^{n-1}$ we have $f\in\Im_{1/4,u}$ implies $Cos_p^{-1}f(v)=0$.*

Proof. For any two pairs of orthogonal unit vectors (ξ,x), (u,v) there exists a rotation $\rho\in SO(n)$ satisfying $u=\rho(x)$, $v=\rho(\xi)$. Since Cos_p^{-1} commutes with rotations, the result follows. □

Lemma 8.3. *Let $n\geqslant 3$, and let $\xi\in x^\perp$. Assume that any $f\in\Im_{1/4,x}$ satisfies $Cos_p^{-1}f(\xi)=0$. Then $Cos_p^{-1}(\Im_{1/2,x})\subset\Im_{1/4,\xi}$.*

Proof. Take any $u \in U_{1/4}(\xi)$. Let $\rho \in SO(n)$, $\rho(\xi) = u$, where ξ is rotated into u inside $U_{1/4}(\xi)$ in the plane containing ξ, u and the origin. Then $\rho(x) \in U_{1/4}(x)$, and $\Im_{1/2,x} \subset \Im_{1/4,\rho(x)}$. Moreover, $Cos_p^{-1} f(u) = 0$ since Cos_p^{-1} commutes with rotations. The point u was chosen arbitrarily in $U_{1/4}(\xi)$, hence $Cos_p^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$. $\square$

Lemma 8.4. *Let $n \geqslant 3$, and let $\xi \in x^{\perp}$. Then there exists a function $f = f_{x,\xi}$ on S^{n-1} satisfying $f_{x,\xi} = 0$ on $U_{1/4}(x)$, but $Cos_p^{-1} f_{x,\xi}(\xi) \neq 0$.*

Proof. Assume the contrary. Then $Cos_p^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$ by Lemma 8.3. Take any vector $y \in S^{n-1}$, and find a vector $q \in x^{\perp} \cap y^{\perp}$. Let $\rho \in SO(n)$ be such that $\rho(x) = x$, $\rho(\xi) = q$. Observe that $f \in \Im_{\epsilon,x}$ implies $f(\rho(\cdot)) \in \Im_{\epsilon,x}$. Since Cos_p^{-1} commutes with rotations, $Cos_p^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$ yields $Cos_p^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,q}$. Take two pairs of orthogonal vectors (x, q) and (q, y). By Lemma 8.2, we have $Cos_p^{-2} f(y) = 0$. Thus, $Cos_p^{-2} f \equiv 0$, a contradiction. $\square$

Lemma 8.5. *Let $n \geqslant 3$. Then there exist an $\varepsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \xi \in S^{n-1}$, there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, and $Cos_p^{-1} f_{x,\xi} \geqslant c$ on $U_\varepsilon(\xi)$.*

Proof. We fix points x and ξ, and provide an $\varepsilon > 0$, and $c > 0$ depending on x, ξ such that there is a function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, and $Cos_p^{-1} f_{x,\xi} \geqslant c > 0$ on $U_\varepsilon(\xi)$. Then we use the compactness argument to prove the statement of the lemma.

Let $\xi \notin x^{\perp}$. Then there exists an $\varepsilon > 0$, such that $\xi \notin E_\varepsilon(x)$. For any function g that is homogeneous of degree p, since p is odd and n is even, by Lemma 7.1 and Eq. (2) the values of $Cos_p(g)$ on $U_\varepsilon(x)$ depend only on the values of g on $E_\varepsilon(x)$. Hence, we may consider an even C^∞-function g such that $g(\pm\xi) > 0$ and $g(\nu) = 0$, for $\nu \in E_\varepsilon(x)$ and define $f_{x,\xi} = Cos_p(g)(x)$.

Let $\xi \in x^{\perp}$. Then Lemma 8.4 implies the existence of $\varepsilon = \varepsilon(x, \xi) = 1/8$, and a function $f = f_{x,\xi}$ on S^{n-1} satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, but $Cos_p^{-1} f_{x,\xi}(\xi) > 0$ (change the sign of $f_{x,\xi}$ if necessary).

Thus, we proved that for any x and ξ, there is $\varepsilon' = \varepsilon'(x, \xi) > 0$ and there is a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $U_{\varepsilon'}(x)$, but $Cos_p^{-1} f_{x,\xi}(\pm\xi) \geqslant c'$, $c' = c'(x, \xi) > 0$. From the continuity of the function $Cos_p^{-1} f_{x,\xi}$ we get that $Cos_p^{-1} f_{x,\xi} \geqslant c$, $c = c(x, \xi) > 0$ on $U_{\varepsilon''}(\xi)$, for some $\varepsilon'' > 0$. Take $\tilde{\varepsilon} = \tilde{\varepsilon}(x, \xi) = \min(\varepsilon', \varepsilon'')$. We show that for any x and ξ, there is $\tilde{\varepsilon} = \tilde{\varepsilon}(x, \xi) > 0$ and there is a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $U_{\tilde{\varepsilon}}(x)$, but $Cos_p^{-1} f_{x,\xi} \geqslant c$ on $U_{\tilde{\varepsilon}}(\xi)$, $c = c(x, \xi) > 0$.

Now we use the compactness argument to prove that we can choose an ε and c independent of x and ξ. We choose a finite set of $\{x_i, \xi_i\}_{i=1}^m$ such that $\{U_{\tilde{\varepsilon}_i/2}(x_i) \times U_{\tilde{\varepsilon}_i/2}(\xi_i)\}_{i=1}^m$ covers $S^{n-1} \times S^{n-1}$. We take

$$\varepsilon = \frac{1}{2} \min_{1 \leqslant i \leqslant m} \tilde{\varepsilon}_i \quad \text{and} \quad c = \min_{1 \leqslant i \leqslant m} c(x_i, \xi_i).$$

Then for any (x, ξ) there is a (x_i, ξ_i) such that

$$U_\varepsilon(x) \times U_\varepsilon(\xi) \subset U_{\tilde{\varepsilon}_i}(x_i) \times U_{\tilde{\varepsilon}_i}(\xi_i),$$

and we may define $f_{x,\xi} = f_{x_i,\xi_i}$. $\square$

Theorem 8.6. *Let $n \geqslant 3$. There exists a convex body K that does not embed in L_p, such that for all $x \in S^{n-1}$ there exists an $\varepsilon(x)$ and a body L_x that does embed in L_p such that $\|\cdot\|_K = \|\cdot\|_{L_x}$ on $U_{\varepsilon(x)}(x)$.*

Proof. Repeat the proof of Theorem 5.7. $\square$

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